Supplementary Material for
“Prototype Discriminative Learning for Image Set Classification”

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I. GRADIENT DERIVATION

In this supplementary material, a detailed derivation is given for the gradient computing formulas of Equ. (8) and (9) in Sec. II-C. As defined in Equ. (5) in Sec. II-B, the objective function can be rewritten as follows.

\[ J(W, P_1, ..., P_C) = \sum_{c=1}^{C} \sum_{x \in X_c} S_{\beta}(Q_x), \]  

(S1)

where

\[ S_{\beta}(z) = \frac{1}{1 + e^{\beta(1-z)}}, \]  

(S2)

and

\[ Q_x = \frac{d(y, nn_{w_i}^c(y))}{d(y, nn_{b_i}^c(y))}. \]  

(S3)

Note that \( d(\cdot, \cdot) \) denotes the Euclidean distance, i.e., for any \( y, z \in \mathbb{R}^r \),

\[ d(y, z) = \sqrt{\sum_{j=1}^{r} (y_j - z_j)^2}. \]  

(S4)
Since we can easily obtain
the variation in
the prototype sets and transformation matrix is sufficiently small. Under such assumption, we can derive
the gradient of loss function $J$ with respect to $W$ approximately according to the Chain Rule:

$$
\frac{\partial J}{\partial W_k} \approx \sum_{c=1}^{C} \sum_{x \in X_c} \frac{S'_{\beta}(Q_x)}{d^2(y, n_{w}^c(y))} \cdot \frac{\partial d(y, n_{w}^c(y))}{\partial W_k} d(y, n_{w}^c(y))
- \sum_{c=1}^{C} \sum_{x \in X_c} \frac{S'_{\beta}(Q_x)}{d^2(y, n_{b}^c(y))} \cdot d(y, n_{w}^c(y)) \frac{\partial d(y, n_{b}^c(y))}{\partial W_k}.
$$

We can easily get

$$
\frac{\partial d(y, n_{w}^c(y))}{\partial W_k} = \frac{2(x - a)(y_k - n_{w}^c(y)_k)}{\sqrt{\sum_{j=1}^{r}(y_j - n_{w}^c(y)_j)^2}} = \frac{(x - a)(y_k - n_{w}^c(y)_k)}{d(y, n_{w}^c(y))},
$$

$$
\frac{\partial d(y, n_{b}^c(y))}{\partial W_k} = \frac{2(x - b)(y_k - n_{b}^c(y)_k)}{\sqrt{\sum_{j=1}^{r}(y_j - n_{b}^c(y)_j)^2}} = \frac{(x - b)(y_k - n_{b}^c(y)_k)}{d(y, n_{b}^c(y))}.
$$

By substituting Equ (S6) into Equ. (S5), Equ. (8) can be finally derived as follows.

$$
\frac{\partial J}{\partial W_k} \approx \sum_{c=1}^{C} \sum_{x \in X_c} \frac{S'_{\beta}(Q_x)}{d^2(y, n_{w}^c(y))} \cdot \frac{d(y, n_{w}^c(y))}{d(y, n_{b}^c(y))} \cdot (x - a)(y_k - n_{w}^c(y)_k)
- \sum_{c=1}^{C} \sum_{x \in X_c} \frac{S'_{\beta}(Q_x)}{d^2(y, n_{b}^c(y))} \cdot \frac{d(y, n_{w}^c(y))}{d(y, n_{b}^c(y))} \cdot (x - b)(y_k - n_{b}^c(y)_k)
= \sum_{c=1}^{C} \sum_{x \in X_c} \frac{S'_{\beta}(Q_x)Q_x}{d^2(y, n_{w}^c(y))} \cdot (x - a)(y_k - n_{w}^c(y)_k)
- \sum_{c=1}^{C} \sum_{x \in X_c} \frac{S'_{\beta}(Q_x)Q_x}{d^2(y, n_{b}^c(y))} \cdot (x - b)(y_k - n_{b}^c(y)_k),
$$

Similarly, we derive the gradient of $J$ with respect to each vector $v_{ci} \in \mathbb{R}^{l_c}$, $i = 1, ..., m_c$, $c = 1, ..., C$.

$$
\frac{\partial J}{\partial v_{ci}} \approx \sum_{c=1}^{C} \sum_{x \in X_c} \frac{S'_{\beta}(Q_x)}{d^2(y, n_{w}^c(y))} \cdot \frac{\partial d(y, n_{w}^c(y))}{\partial v_{ci}} d(y, n_{w}^c(y))
- \sum_{c=1}^{C} \sum_{x \in X_c} \frac{S'_{\beta}(Q_x)}{d^2(y, n_{b}^c(y))} \cdot d(y, n_{w}^c(y)) \frac{\partial d(y, n_{b}^c(y))}{\partial v_{ci}}.
$$

Since we can easily obtain

$$
\frac{\partial d(y, n_{w}^c(y))}{\partial v_{ci}} = U_c^T W W^T (a - x),
$$

$$
\frac{\partial d(y, n_{b}^c(y))}{\partial v_{ci}} = U_c^T W W^T (b - x),
$$

we can finally deduce Equ. (9)
**Algorithm 1** Optimization algorithm for PDL-OP

**Input:**

- Data matrices of \( C \) image sets for training: \( \{X_1, X_2, \ldots, X_C\} \) and their labels;
- the slope for sigmoid function: \( \beta \);
- the initial prototype sets: \( P = \{P_1, \ldots, P_C\} \);
- the initial transformation matrix: \( W \).

**Output:**

- The optimal \( P \) and \( W \)

1: \textbf{while} not converged \textbf{do}

2: Compute the loss function according to Equ. (5);

3: Generate \( A \) according to Equ. (S10);

4: Compute the step size \( \tau \) for searching \( W \) by the curvilinear searching algorithm [1]. Call line search along the path \( W(\tau) \) defined by Equ. (S11);

5: Compute the update of \( V \) by the L-BFGS algorithm;

6: Update \( V \) and \( W \);

7: \textbf{end while}

8: Compute \( P \) by affine coefficients \( V \) according to Equ. (3);

9: \textbf{return} \( P^*, W^* \);

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**II. OPTIMIZATION ALGORITHM**

To solve the optimization problem \( \min_{W,V} J(W,V) \) without projection constraint, we can update \( W \) and \( V \) in an iterative procedure based on the derived gradients above by using limited-memory BFGS (L-BFGS) [2] to control the step size.

As for the optimization problem \( \min_{W,V} J(W,V) \) with orthonormal projection constraint \( W^T W = I_r \), it is non-trivial to jointly optimize \( W \) and \( V \) as the feasible set of \( W \) is on a Stiefel manifold. Considering its favorable property of low computational cost, we choose the curvilinear searching method in [1] to search the step size and the path of \( W \).

Formally, according to [1], we define a skew-symmetric matrix \( A \) as

\[
A = GW^T - WG^T.
\]  

(S10)

where \( G = \frac{\partial J}{\partial W} \). To drive the new point to satisfy \( W'^T W' = I_r \), it is searched along a curve \( W(\tau) \) given
by

\[ W(\tau) = W - \tau A \left( \frac{W + W(\tau)}{2} \right), \]  

(S11)

where \( \tau \) is a step size and \( W(\tau) \) satisfies \( W(\tau)^T W(\tau) = W^T W \).

Furthermore, to guarantee the optimization of \( W \) and \( V \) to be performed jointly and consistently, at each step, we update \( W \) along the curve \( W(\tau) \) by using [1] and search the new iteration of \( V \) along straight lines by using L-BFGS. The optimization algorithm is summarized in Alg. 1.

REFERENCES
